Thorin Quantisation

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Abstract

The reciprocal of a normalised Laguerre–Pólya entire function is the Laplace transform of a generalised gamma convolution (GGC) whose Thorin measure is supported on the squared zeros. We show that for entire functions of genus zero the converse holds in a strong form: if the associated clock is a GGC at all, its Thorin measure is *automatically* a counting measure with unit atoms. Quantisation is therefore not an extra hypothesis but a consequence, and the entire content of the Riemann Hypothesis, read through the van Dantzig–Wald dictionary, sits in the single step from infinite divisibility to the Thorin class. We make this precise in three ways. First, we identify the GGC locus inside the class of van Dantzig solutions constructed by Konstantopoulos, Patie and Sarkar from Laplace exponents of spectrally negative Lévy processes: their hitting-time clocks are always self-decomposable, and are GGC exactly when the associated entire function has only real zeros; the Mittag-Leffler instances give explicit self-decomposable hitting times that are not GGC. Second, we show that no Lévy–Khintchine representation of $\log \xi(\alpha + s)$ with a non-negative Lévy measure can be established at $\alpha = \frac{1}{2}$ without already assuming the absence of zeros to the right of α, and we correct a spurious Thorin atom that arises when ξ is factored through the pole of ζ. Third, working from the shifted basepoint $s = \frac{3}{2}$, we prove that RH is equivalent to $\sigma \mapsto \xi(\frac{3}{2})/\xi(\frac{1}{2} + \sqrt{1+\sigma})$ being the Laplace transform of a GGC, equivalently to its hyperbolic complete monotonicity, and we convert this into a determinantal criterion: RH holds if and only if the Hankel matrices of the power sums $c_n = \sum_\rho [(\frac{3}{2} - \rho)(\rho + \frac{1}{2})]^{-(n+1)}$, summed over zero pairs, are positive semidefinite. The criterion is unconditionally computable from the Taylor coefficients of ξ at $\frac{3}{2}$; we verify it numerically through order five and recover $\gamma_1 = 14.1347\ldots$ from the coefficient ratios.

Keywords. Riemann hypothesis, generalised gamma convolution, Thorin measure, van Dantzig pair, Wald couple, Laguerre–Pólya class, Thorin–Bernstein function, hyperbolically completely monotone, Hankel determinant, self-decomposability.
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1 Introduction

Let f be an even entire function, real on $\mathbb{R}$, normalised by $f(0) = 1$. Two classical toolkits address the question of whether the zeros of f are all real. The Riemann–Pólya approach works with f directly, through Fourier integrals, universal factors and the Laguerre–Pólya (LP) class. The Thorin–Bondesson approach works with $1/f$, as the Laplace transform of a positive random variable, and exploits the fact that Laplace transforms of generalised gamma convolutions (GGC) are analytic and zero-free on the cut plane $\mathbb{C}\setminus(-\infty,0]$. In [20] the author developed the duality between the two: the Hadamard–Weierstrass factorisation of an LP function is dual to a van Dantzig pair of characteristic functions [28, 14] and to a Wald couple of infinitely divisible random variables [23], and Thorin's condition — a real Laplace identity on $(0,\infty)$ — encodes the absence of zeros off the critical axis.

Konstantopoulos, Patie and Sarkar [10] have since carried out a deep study of the van Dantzig problem itself. Two of their results reshape the programme. They prove that the Lukacs class $\mathbb{D}_L$ of even LP characteristic functions is a *proper* subset of the van Dantzig class $\mathbb{D}$, exhibiting the first solutions of van Dantzig's problem with non-real zeros; and they construct a large new family $\mathbb{D}_P\subset\mathbb{D}$, indexed by Laplace exponents of spectrally negative Lévy processes, in which both members of each pair are identified probabilistically — one as the hitting time of a self-similar Markov process, the other as an arcsine-type variable multiplied by the square root of the exponential functional of a subordinator. The first result is a warning: positive definiteness of $1/\xi(it)$ is strictly weaker than RH. The second is an opportunity: a laboratory of explicit clocks on which to test where exactly the strength lies.

This paper works on that boundary. Our results are of four kinds.

Automatic quantisation. Section 3 proves a rigidity statement (Theorem 3.1). Let g be entire of order less than one with $g(0)=1$ and real Taylor coefficients. Then the following are equivalent: all zeros of g are real and negative; $1/g$ is the Laplace transform of a GGC; $1/g$ is hyperbolically completely monotone; $-\log(1/g)$ is a Thorin–Bernstein function. Moreover, whenever these hold, the Thorin measure is forced to be $\sum_k\delta_{a_k}$, a counting measure with *unit* atoms at the zero moduli. This is the probabilistic content of Kwaśnicki's integer-valued characterisation of the Lukacs class [11]: real zeros correspond to quantised Thorin mass, and for genus-zero functions the quantisation cannot be avoided. In particular one cannot weaken the target from "GGC with unit atoms" to "GGC": for these functions the two are the same statement.

The GGC locus inside the Patie class. Section 4 locates the boundary explicitly. For Ψ in the admissible set of [10] let T_Ψ be the associated hitting-time clock, so that $\mathbb{E}(e^{-uT_\Psi})=1/\mathcal{I}_\Psi(\sqrt{u})=e^{-\phi_\Psi(u)}$. Theorem 4.1 states that $\mathcal{J}_\Psi\in\mathbb{D}_L$ if and only if ϕ_Ψ is Thorin–Bernstein, if and only if T_Ψ is a GGC. Since T_Ψ is *always* self-decomposable [17], the gap $\mathbb{D}_L\subsetneq\mathbb{D}$ is exactly the gap GGC $\subsetneq$ SD between the Thorin class and self-decomposability. Corollary 4.3 makes this concrete: for $1<\alpha<2$ the Mittag-Leffler exponents produce hitting times that are self-decomposable but not GGC. To our knowledge these are the first explicit such hitting times in the literature. Proposition 4.6 records the growth constraint that any Thorin measure attached to ξ must satisfy, $U([0,T^2])\sim\frac{T}{2\pi}\log\frac{T}{2\pi e}$, which forces the Blumenthal–Getoor regime $\Psi(u)\asymp u^2/\log u$ isolated in [10].

Two obstructions on the critical line. Section 5 is a cautionary analysis of the temptation to push a Lévy representation of $\log\xi(\alpha+s)$ from $\alpha>1$ down to $\alpha=\frac{1}{2}$. Proposition 5.1 shows that the existence of such a representation with a non-negative Lévy measure at α already *implies* the absence of zeros of ζ in $\mathrm{Re}(s)>\alpha$; at $\alpha=\frac{1}{2}$ it is therefore not a step towards RH but a restatement of it, and continuation in α is circular, and any successful argument must transfer information from $\alpha>1$ by some other mechanism. Lemma 5.3 isolates a second, more mundane trap: factoring ξ as (linear)$\times$(gamma)$\times\zeta$ produces a spurious Thorin atom at the point where the pole of ζ cancels the linear zero, and the cure is to work with the de-poled form $\xi(s) = \pi^{-s/2}\Gamma(1 + \frac{s}{2})(s-1)\zeta(s)$.

A shifted basepoint and a determinantal criterion. Sections 6 and 7 contain the main results. Write $G(v) = \xi(\frac{1}{2} + \sqrt{v})$, an entire function of v of order $\frac{1}{2}$, and for $c \geq 0$ set $\Xi_c(\sigma) = G(c+\sigma)/G(c)$. Theorem 6.3 states that RH holds if and only if $1/\Xi_c$ is the Laplace transform of a GGC, for one and then every $c \geq 0$, in which case the Thorin measure is $\sum_{\gamma>0} \delta_{c+\gamma^2}$. Because ξ has no real zeros, the shifted basepoint costs nothing in strength while removing the pole and exponential-tilting artefacts of Section 5: the Thorin measure is bounded away from the origin, all exponential moments below $c + \gamma_1^2 \approx 200.8$ exist, and the clock H_c is the Esscher transform of the classical one by e^{-cH_0}.

Theorem 7.2 then converts hyperbolic complete monotonicity into positivity of explicit determinants. Let

$$c_n \;=\; \sum_{\rho} \Big[\big(\tfrac{3}{2} - \rho\big)\big(\rho + \tfrac{1}{2}\big)\Big]^{-(n+1)}, \qquad n \geq 0,$$

the sum running over zero pairs $\{\rho, 1-\rho\}$ of ζ. These are unconditionally computable from the Taylor coefficients of ξ at $s = \frac{3}{2}$. Then RH holds if and only if the Hankel matrices $[c_{i+j}]_{0 \leq i,j \leq N}$ and $[c_{i+j+1}]_{0 \leq i,j \leq N}$ are positive semidefinite for every N. This is a Stieltjes-moment criterion, comparable in shape to Li's criterion [13] and to the Jensen-polynomial hyperbolicity of [7], but living on the reciprocal side of the duality. Section 8 reports a numerical verification through $N = 5$ at sixty digits, together with the pleasant observation that $c_{n+1}/c_n \to (1 + \gamma_1^2)^{-1}$, so that the first zero ordinate is recoverable from the moment sequence to six digits by order thirteen.

Section 10 lists what remains. The programme now has a single target: prove the Hankel positivity of Theorem 7.2, or equivalently the hyperbolic complete monotonicity of Theorem 6.3, unconditionally.

2 Preliminaries

Throughout, all random variables are real-valued and defined on a common probability space when convenient; only their laws enter.

2.1 Entire functions

Definition 2.1. An entire function g belongs to the Laguerre–Pólya class $\mathcal{LP}$ if it is real on $\mathbb{R}$ and is the local uniform limit of real polynomials with only real zeros; equivalently

$$g(z) = Kz^m e^{-c^2z^2+az} \prod_{k\geq 1} \Big(1 - \frac{z}{z_k}\Big) e^{z/z_k}, \qquad z_k \in \mathbb{R} \setminus \{0\},\ \sum_k z_k^{-2} < \infty.$$

We write $\mathcal{LP}_e$ for the even elements normalised by $g(0) = 1$, which take the form $g(z) = e^{-c^2z^2} \prod_k (1 - z^2/z_k^2)$.

If g is entire of order $\varrho < 1$ with $g(0) = 1$ then g has genus zero and Hadamard's theorem gives $g(z) = \prod_k (1 - z/a_k)$ with $\sum_k |a_k|^{-1} < \infty$ and no exponential factor. We use this repeatedly with $\varrho = \frac{1}{2}$.

2.2 van Dantzig pairs and Wald couples

Let $\mathbb{P}_+$ be the set of continuous positive definite functions on $\mathbb{R}$ normalised to 1 at the origin, i.e. by Bochner's theorem the set of characteristic functions. Van Dantzig's problem [28, 14] asks for a description of

$$\mathbb{D} = \{\mathcal{F} \in \mathbb{P}_+ : \ V\mathcal{F} \in \mathbb{P}_+\}, \qquad V\mathcal{F}(t) = \frac{1}{\mathcal{F}(it)},$$

and $[\mathcal{F}, V\mathcal{F}]$ is called a van Dantzig pair. The Lukacs class is $\mathbb{D}_L = \mathcal{LP}_e \cap \mathbb{P}_+ \subset \mathbb{D}$. Two facts frame everything below.

Theorem 2.2 (Schoenberg [25])**.** *If $g \in \mathcal{LP}_e$ then $t \mapsto 1/g(it)$ is the characteristic function of a symmetric Pólya frequency density; that density is infinitely divisible [11].*

Definition 2.3. Random variables X on $\mathbb{R}$ and $H \geq 0$ form a Wald couple if $\mathbb{E}(e^{sX})\,\mathbb{E}(e^{-s^2H}) = 1$ for s in a real neighbourhood of the origin.

If $\mathcal{F} \in \mathbb{D}$ and Y has characteristic function $V\mathcal{F}$, the defining identity $\mathcal{F}(it)V\mathcal{F}(t) = 1$ says exactly that $\mathbb{E}(e^{sY}) = 1/\mathcal{F}(s)$; when $Y = \sqrt{2}\,B_H$ is a Brownian motion run to an independent clock H one recovers $\mathbb{E}(e^{s\sqrt{2}B_H}) = \mathbb{E}(e^{s^2H})$ and hence a Wald couple. Lukacs' rigidity theorem [14] states that if both members of a van Dantzig pair are infinitely divisible then both are Gaussian; generic pairs are therefore asymmetric, with the clock side infinitely divisible and the other side not.

2.3 Generalised gamma convolutions

Definition 2.4 (Thorin [26], Bondesson [4])**.** A random variable $H \geq 0$ is a generalised gamma convolution, $H \in \mathrm{GGC}$, if there are $a \geq 0$ and a σ-finite measure U on $(0, \infty)$ with $\int_{(0,1)} |\log z| U(dz) < \infty$ and $\int_{[1,\infty)} z^{-1}U(dz) < \infty$, such that

$$\mathbb{E}(e^{-\sigma H}) = \exp\Big(-a\sigma - \int_{(0,\infty)} \log\Big(1 + \frac{\sigma}{z}\Big)\,U(dz)\Big), \qquad \sigma > 0. \tag{1}$$

U is the Thorin measure of H.

The class GGC is the smallest class of laws on $[0, \infty)$ containing the gamma laws and closed under independent sums and weak limits. We use four standard facts [4].

Theorem 2.5 (Analyticity)**.** *If $H \in \mathrm{GGC}$ then $\sigma \mapsto \mathbb{E}(e^{-\sigma H})$ extends to a holomorphic and zero-free function on the cut plane $\mathbb{C} \setminus (-\infty, 0]$.*

Theorem 2.6 (Uniqueness)**.** *The map $H \mapsto (a, U)$ is a bijection onto the admissible pairs.*

Theorem 2.7 (Hyperbolic complete monotonicity)**.** *A function $h : (0,\infty) \to (0,\infty)$ is* HCM *if for every $u > 0$ the map $w \mapsto h(uv)h(u/v)$ is completely monotone in $w = v + v^{-1}$. The Laplace transform of a GGC is* HCM*, and conversely an* HCM *function which is the Laplace transform of a probability measure on $[0, \infty)$ is the Laplace transform of a GGC.*

Definition 2.8. $\Phi : (0, \infty) \to [0, \infty)$ is a Thorin–Bernstein function, $\Phi \in \mathcal{TBF}$, if $\Phi(\sigma) = a\sigma + \int \log(1 + \sigma/z)U(dz)$ as in (1); equivalently Φ is Bernstein and Φ' is a Stieltjes function,

$$\Phi'(\sigma) = a + \int_{(0,\infty)} \frac{U(dz)}{\sigma + z}. \tag{2}$$

One has $\mathcal{TBF} \subsetneq \mathcal{CBF} \subsetneq \mathcal{B}$, where $\mathcal{CBF}$ denotes the complete Bernstein (Pick) functions [24].

Finally recall the inclusions, for laws on $[0,\infty)$,

$$\mathrm{GGC} \subsetneq \mathrm{SD} \subsetneq \mathrm{ID},$$

SD denoting self-decomposability. The content of this paper is that the first inclusion is the one that carries arithmetic information.

3 Rigidity: Thorin measures of genus-zero functions are quantised

The following theorem is elementary but is the organising principle of the paper. It says that for entire functions of genus zero one cannot ask for less than quantisation: if the clock is Thorin at all, its Thorin measure is a counting measure with unit atoms.

Theorem 3.1. *Let g be entire of order $\varrho < 1$, real on $\mathbb{R}$, with $g(0) = 1$, and let $\{-a_k\}_{k\geq 1}$ be its zeros, repeated with multiplicity (so that a_k is minus the k-th zero). The following are equivalent.*

(i) $a_k \in (0,\infty)$ *for every k, i.e.* $g(\sigma) = \prod_k (1 + \sigma/a_k)$ *with* $a_k > 0$ *and* $\sum_k a_k^{-1} < \infty$.

(ii) $1/g$ *is the Laplace transform of a random variable* $H \geq 0$ *which is a GGC.*

(iii) $1/g$ *is* HCM *on* $(0,\infty)$.

(iv) $\log g$ *is a Thorin–Bernstein function.*

Moreover, when these hold, $H \overset{d}{=} \sum_k \mathbf{e}_k/a_k$ with $\mathbf{e}_k$ independent standard exponentials, the drift vanishes, and the Thorin measure is the counting measure

$$U_g = \sum_{k\geq 1} \delta_{a_k},$$

uniquely determined by g. In particular the atoms of U_g all have mass one (more precisely, mass equal to the multiplicity of the corresponding zero).

Proof. (i)⇒(ii),(iv) and the final statements. If $a_k > 0$ and $\sum a_k^{-1} < \infty$ then $1/g(\sigma) = \prod_k (1+\sigma/a_k)^{-1}$ is, factor by factor, the Laplace transform of an $\mathrm{Exp}(a_k)$ variable; the product converges and, the summands being non-negative with $\mathbb{E}\sum_k \mathbf{e}_k/a_k = \sum_k a_k^{-1} < \infty$, the series converges almost surely by monotone convergence, so $1/g = \mathbb{E}(e^{-\sigma H})$ with $H = \sum_k \mathbf{e}_k/a_k$. Since $\log g(\sigma) = \sum_k \log(1+\sigma/a_k) = \int \log(1+\sigma/z) U_g(dz)$ with $U_g = \sum_k \delta_{a_k}$, and the integrability conditions $\int_{(0,1)} |\log z| U_g(dz) < \infty$, $\int_{[1,\infty)} z^{-1} U_g(dz) < \infty$ follow from $\sum_k a_k^{-1} < \infty$ together with the fact that only finitely many a_k lie in $(0,1)$, we get $H \in \mathrm{GGC}$ with Thorin measure U_g and zero drift.

(ii)⇒(i). Suppose $1/g = \mathbb{E}(e^{-\sigma H})$ with $H \in \mathrm{GGC}$. By Theorem 2.5 the right-hand side is holomorphic and zero-free on $\mathbb{C} \setminus (-\infty, 0]$. Hence $1/g$, a priori meromorphic with poles exactly at the zeros of g, has no poles in $\mathbb{C} \setminus (-\infty, 0]$; equivalently g has no zeros there. Since $g(0) = 1$, all zeros lie in $(-\infty, 0)$, i.e. $a_k > 0$. As $\varrho < 1$, Hadamard's factorisation theorem gives $g(\sigma) = \prod_k (1+\sigma/a_k)$ with $\sum_k a_k^{-1} < \infty$, which is (i). Uniqueness of the Thorin measure (Theorem 2.6) identifies $U = U_g$.

(ii)⇔(iii) is Theorem 2.7, and (ii)⇔(iv) is the definition (1) combined with Theorem 2.6. □

Remark 3.2 (Why quantisation is free)**.** Condition (ii) makes no reference to atoms; the atoms are produced by the proof. The mechanism is that an entire function of genus zero has no room for a diffuse Thorin measure: a diffuse U would make $\exp(-\int \log(1+\sigma/z)U(dz))$ non-meromorphic. This is the analytic shadow of Kwaśnicki's theorem [11, Prop. 5.3] that membership of $\mathbb{D}_L$ forces the density ρ in the Lévy–Khintchine type representation to be integer-valued.

Corollary 3.3 (Wald form)**.** *Let g be as in Theorem 3.1 and set $f(s) = g(-s^2)$, an even entire function of order $2\varrho < 2$ with $f(0) = 1$, whose zeros are $\pm\sqrt{a_k}$.*

(a) If the equivalent conditions of Theorem 3.1 hold then $f \in \mathcal{LP}_e$; the function $s \mapsto 1/f(is) = 1/g(s^2) = \mathbb{E}(e^{-s^2 H})$ is the characteristic function of $\sqrt{2}B_H$ with $H \in \mathrm{GGC}$; $(\sqrt{2}B_H, 2H)$ is a Wald couple; and the Thorin measure is $\sum_k \delta_{a_k}$, the counting measure on the squared zeros of f.

(b) Conversely, if $f \in \mathbb{P}_+$ and $[f, 1/f(i\cdot)]$ is a van Dantzig pair whose clock is a GGC, then f has only real zeros.

Proof. Both parts follow from Theorem 3.1 with $\sigma = s^2$, Theorem 2.2 for the identification of $1/f(is)$ as a characteristic function, and the Gaussian mixture identity $\mathbb{E}(e^{isB_{2H}}) = \mathbb{E}(e^{-s^2 H})$. □

Remark 3.4. The asymmetry between (a) and (b) is not an artefact. Membership of $\mathcal{LP}_e$ does *not* imply positive definiteness: as emphasised in [10, §2], an even entire function of the form $e^{-c^2z^2}\prod_k(1 - z^2/z_k^2)$ need not be a characteristic function. Schoenberg's theorem delivers only the reciprocal side. Consequently, statements of the form "f and $1/f(i\cdot)$ form a van Dantzig pair" require $f \in \mathbb{P}_+$ as a separate input, which for ξ is supplied by the Fourier integral $\xi(\frac{1}{2} + it) = \int_{\mathbb{R}} e^{itx}\Phi(x)dx$ and is not available for the shifted functions of Section 6.

Remark 3.5 (Where the strength is)**.** Corollary 3.3 should be read against the hierarchy GGC $\subsetneq$ SD $\subsetneq$ ID. Membership of $\mathbb{D}$ only says that the clock exists; Schoenberg's theorem upgrades $\mathbb{D}_L$-membership to infinite divisibility of the clock; but infinite divisibility, and even self-decomposability, are strictly weaker than the Thorin property, and it is only the Thorin property that returns real zeros. Section 4 makes this quantitative by exhibiting the gap.

4 The GGC locus inside the Patie class

We recall the construction of [10]. Let

$$\Psi(u) = -\kappa + au + \tfrac{1}{2}\sigma^2u^2 - \int_0^\infty \big(1 - e^{-ur} - ur\mathbf{1}_{\{r<1\}}\big)\mu(dr)$$

be the Laplace exponent of a possibly killed spectrally negative Lévy process, $\theta = \sup\{u \geq 0 : \Psi(u) = 0\}$, and let $\mathbb{N}_{\mathbb{D}} = \{\Psi : 0 \leq \theta \leq \frac{1}{2}\}$. Put

$$\mathcal{J}_\Psi(t) = \sum_{n\geq 0} \frac{(-1)^n}{W_\Psi(n+1)} t^{2n}, \qquad W_\Psi(n+1) = \prod_{k=1}^{n} \Psi(k), \qquad \mathcal{I}_\Psi(t) = \mathcal{J}_\Psi(it),$$

and $\mathbb{D}_P = \{\mathcal{J}_\Psi : \Psi \in \mathbb{N}_\mathbb{D}\}$. By [10, Thm. 20], $[\mathcal{J}_\Psi, 1/\mathcal{I}_\Psi]$ is a van Dantzig pair, and the clock side is identified: with $X(\Psi)$ the 1-self-similar Markov process associated with Ψ through Lamperti's transform and $T_\Psi = \inf\{t > 0 : X_t(\Psi) \geq 1\}$ started from 0,

$$\mathbb{E}\big(e^{-uT_\Psi}\big) = \frac{1}{\mathcal{I}_\Psi(\sqrt{u})} = e^{-\phi_\Psi(u)}, \qquad u > 0, \tag{3}$$

ϕ_Ψ being a Bernstein function, and T_Ψ is self-decomposable [17]. Moreover $\mathcal{J}_\Psi$ is the characteristic function of $\sqrt{I_\phi} \times J_\theta$, where $I_\phi = \int_0^\infty e^{-Z_t} dt$ is the exponential functional of the subordinator with exponent ϕ arising from the Wiener–Hopf factorisation $\Psi(u) = (u - \theta)\phi(u)$, and J_θ is an independent symmetric Beta-type variable.

Theorem 4.1. *Let $\Psi \in \mathbb{N}_\mathbb{D}$ and let T_Ψ be its clock (3). The following are equivalent.*

(i) $\mathcal{J}_\Psi \in \mathbb{D}_L$, *i.e.* $\mathcal{J}_\Psi$ *has only real zeros.*

(ii) $T_\Psi \in \mathrm{GGC}$.

(iii) $\phi_\Psi \in \mathcal{TBF}$.

(iv) $u \mapsto 1/\mathcal{I}_\Psi(\sqrt{u})$ *is* HCM.

When they hold, the Thorin measure of T_Ψ is $\sum_k \delta_{z_k^2}$, the counting measure on the squared zeros of $\mathcal{J}_\Psi$, and $\mathcal{J}_\Psi(t) = \prod_k (1 - t^2/z_k^2)$.

Proof. Write $g_\Psi(u) = \mathcal{I}_\Psi(\sqrt{u}) = \sum_{n\geq 0} u^n / W_\Psi(n+1)$. Since $\Psi(n) \to \infty$, g_Ψ is entire; by [1, Prop. 2.1], see also [10, Prop. 31], its order is $\varrho_\Psi = 1/\underline{\Psi} \in [\frac{1}{2}, 1]$ where $\underline{\Psi} \in [1, 2]$ is the Blumenthal–Getoor lower index of Ψ; in particular $\varrho_\Psi \leq 1$. If $\varrho_\Psi < 1$ we may apply Theorem 3.1 to g_Ψ directly, and (i)–(iv) are its conditions (i)–(iv) transported through $u = t^2$, using $\mathcal{J}_\Psi(t) = \mathcal{I}_\Psi(it) = g_\Psi(-t^2)$.

It remains to treat the boundary case $\varrho_\Psi = 1$, where Hadamard's theorem allows a factor e^{bu}. The implication (i)⇒(ii) is unaffected: if $\mathcal{J}_\Psi \in \mathbb{D}_L$ then $\mathcal{J}_\Psi(t) = e^{-ct^2} \prod_k (1 - t^2/z_k^2)$ with $c \geq 0$, hence $g_\Psi(u) = e^{cu} \prod_k (1 + u/z_k^2)$ and $\phi_\Psi(u) = cu + \sum_k \log(1 + u/z_k^2) \in \mathcal{TBF}$, so $T_\Psi \in \mathrm{GGC}$ with drift c and Thorin measure $\sum_k \delta_{z_k^2}$. Conversely if $T_\Psi \in \mathrm{GGC}$ then by Theorem 2.5 $1/g_\Psi$ is zero-free and holomorphic on $\mathbb{C}\backslash(-\infty, 0]$, so g_Ψ has all its zeros in $(-\infty, 0)$; moreover the GGC integrability condition $\int_{[1,\infty)} z^{-1} U(dz) < \infty$ forces $\sum_k a_k^{-1} < \infty$ for the zeros $-a_k$, so the Weierstrass exponential factors are unnecessary and Hadamard's theorem for order one gives $g_\Psi(u) = e^{bu} \prod_k (1 + u/a_k)$ with $a_k > 0$ and $b \geq 0$, whence $\mathcal{J}_\Psi(t) = e^{-bt^2} \prod_k (1 - t^2/a_k)$ has only real zeros. The equivalences with (iii) and (iv) are as before. □

Corollary 4.2 (The gap is exactly GGC versus SD)**.** $\mathbb{D}_P \cap \mathbb{D}_L = \{\mathcal{J}_\Psi : \phi_\Psi \in \mathcal{TBF}\}$. *Since T_Ψ is self-decomposable for every $\Psi \in \mathbb{N}_\mathbb{D}$, and since $\mathbb{D}_L \subsetneq \mathbb{D}$ by [10, Thm. 23], there exist $\Psi \in \mathbb{N}_\mathbb{D}$ whose clocks are self-decomposable but not GGC.*

We can name them.

Corollary 4.3 (Mittag-Leffler clocks are not Thorin)**.** *Let $1 < \alpha < 2$ and $\beta \in (\alpha - 1, \alpha)$, and let*

$$\Psi(u) = \frac{\Gamma(\alpha u + \beta)}{\Gamma(\alpha u + \beta - \alpha)} \in \mathbb{N}_\mathbb{D}, \qquad \textit{so that} \qquad \mathcal{J}_\Psi(t) = \Gamma(\beta) E_{\alpha,\beta}(-t^2).$$

Then T_Ψ is self-decomposable but is not a generalised gamma convolution; equivalently, ϕ_Ψ is Bernstein but not Thorin–Bernstein, and $u \mapsto 1/\mathcal{I}_\Psi(\sqrt{u})$ is completely monotone but not HCM.

Ψ	$\mathcal{J}_\Psi$	zeros	clock T_Ψ
$u(u+\nu)$, $\nu \geq -\frac{1}{2}$	Bessel–Clifford	real	GGC
$u\Gamma(\alpha u+\beta)/\Gamma(\alpha(u-1)+\beta)$	Fox–Wright	real	GGC
$u\frac{u+1-a}{u+b}$, $0<a<1$, $a+b\in\mathbb{Z}_{\geq 2}$	${}_1F_1$	real	GGC
$\Gamma(\alpha u+\beta)/\Gamma(\alpha u+\beta-\alpha)$, $1<\alpha<2$	Mittag-Leffler	non-real	SD \ GGC
$u(u+\gamma)^\alpha$, $\alpha\in(0,1)$	power-gamma	open	open

Table 1: The Thorin locus inside the Patie class. Membership of the last column is equivalent to reality of the zeros by Theorem 4.1.

Proof. The identification of $\mathcal{J}_\Psi$ with the Mittag-Leffler function is [10, §4.2.4]. For $1<\alpha<2$ the function $E_{\alpha,\beta}$ has non-real zeros [6, 21], so $\mathcal{J}_\Psi \notin \mathbb{D}_L$; apply Theorem 4.1. Self-decomposability is [17]. □

Corollary 4.3 is the falsification test for the whole framework, and the framework passes it: a clock that is merely self-decomposable does not deliver real zeros, and the Thorin condition detects the failure. It also supplies, we believe for the first time, explicit hitting times of self-similar Markov processes that are self-decomposable and provably not GGC — in contrast to Kent's theorem [9] that hitting times of one-dimensional diffusions are always GGC. The Mittag-Leffler processes are of course not diffusions; the corollary quantifies how far the jump case departs from Kent's picture.

Example 4.4 (Bessel; Kent's theorem recovered)**.** Let $\Psi(u)=u(u+\nu)$, $\nu\geq-\frac{1}{2}$, so $\theta=0$, $\phi(u)=u+\nu$ and $\mathcal{J}_\Psi(t)=\Gamma(\nu+1)t^{-\nu}J_\nu(2t)$. Then $\phi_\Psi(u)=\sum_k \log\left(1+4u/j_{\nu,k}^2\right)\in\mathcal{TBF}$ and $T_\Psi\in\mathrm{GGC}$ with Thorin measure $\sum_k \delta_{j_{\nu,k}^2/4}$, in agreement with [8, 9]. All the classical van Dantzig pairs $[\cos t, 1/\cosh t]$, $[\sin t/t,\ t/\sinh t]$ are the cases $\nu=\mp\frac{1}{2}$.

Example 4.5 (Fox–Wright)**.** For $\alpha\in(0,1)$, $\beta\geq\alpha$ and $\Psi(u)=u\Gamma(\alpha u+\beta)/\Gamma(\alpha(u-1)+\beta)$ one has $\mathcal{J}_\Psi={}_0\Psi_1^*((\beta,\alpha);-t^2)\in\mathbb{D}_L$ by Laguerre's theorem, since $z\mapsto 1/\Gamma(\alpha z+\beta)$ has only real negative zeros. Hence the corresponding clock is GGC by Theorem 4.1, with Thorin measure the counting measure on the squared zeros of the normalised Fox–Wright function.

We close the section with the growth constraint that any candidate representation of ξ within this framework must respect.

Proposition 4.6. *Assume RH and let H_ξ be the clock of ξ, i.e. $\xi(\frac{1}{2})/\xi(\frac{1}{2}+t)=\mathbb{E}(e^{-t^2H_\xi})$. Then the Thorin measure of H_ξ is $U_\xi=\sum_{\gamma>0}\delta_{\gamma^2}$, with counting function*

$$U_\xi\big([0,T^2]\big)=N(T)\sim\frac{T}{2\pi}\log\frac{T}{2\pi e},$$

and consequently $\phi_\xi(t^2)=\log\big(\xi(\frac{1}{2}+t)/\xi(\frac{1}{2})\big)\sim\frac{t}{2}\log t$ as $t\to\infty$. In particular ξ has order one and maximal type, so if $\xi=\mathcal{J}_\Psi$ for some $\Psi\in\mathbb{N}_{\mathbb{D}}$ then Ψ has Blumenthal–Getoor index $\underline{\Psi}=2$ and $\mathcal{J}_\Psi$ has infinite type. If moreover Ψ is regularly varying then necessarily $\Psi(u)=u^2/\ell(u)$ with ℓ slowly varying and $\ell(u)\to\infty$, the regime of [10, Prop. 26]. In particular no Ψ with a Gaussian component $\sigma^2>0$ can occur, since then $\Psi(u)\sim\frac{1}{2}\sigma^2u^2$, the type is finite, and maximal type fails.

Proof. The identification of U_ξ is Corollary 3.3. For the asymptotics, smooth N to the density $U_\xi(dz) \approx \frac{\log z}{8\pi\sqrt{z}}dz$ and use $\int_0^\infty z^{s-1}\log(1+u/z)dz = u^s\pi/(s\sin\pi s)$ for $0<s<1$, which at $s=\frac{1}{2}$ gives $\int_0^\infty \log(1+u/z)z^{-1/2}dz = 2\pi\sqrt{u}$, together with its derivative in s, $\int_0^\infty \log(1+u/z)z^{-1/2}\log z\, dz \sim 2\pi\sqrt{u}\log u$, to obtain $\phi_\xi(u) \sim \frac{1}{4}\sqrt{u}\log u$, i.e. $\phi_\xi(t^2) \sim \frac{t}{2}\log t$. This matches $\log\xi(\frac{1}{2}+t) \sim \frac{t}{2}\log t$, which follows from Stirling applied to $\Gamma(1+\frac{s}{2})$ in $\xi(s) = \pi^{-s/2}\Gamma(1+\frac{s}{2})(s-1)\zeta(s)$. For the final statements, $\varrho_\Psi = 2/\underline{\Psi} = 1$ forces $\underline{\Psi} = 2$; the type formula of [10, Prop. 31] gives $\tau_\Psi \geq (\limsup_n n/\phi(n))^{1/2}$, so maximal type forces $\limsup_n n/\phi(n) = \infty$, which fails when $\sigma^2 > 0$ since then $\phi(n) \sim \frac{1}{2}\sigma^2 n$. The regularly varying case is [10, Prop. 26]. $\square$

5 Two obstructions on the critical line

The natural strategy for ξ is to build a Lévy representation of $\log\xi(\alpha+s)$ from the Euler product where it converges, and to lower α to $\frac{1}{2}$. This section shows what stands in the way. Throughout

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-s/2}\,\Gamma\big(1+\tfrac{s}{2}\big)\,(s-1)\zeta(s), \tag{4}$$

using $\frac{s}{2}\Gamma(\frac{s}{2}) = \Gamma(1+\frac{s}{2})$. Recall that ξ is entire of order one, $\xi(s) = \xi(1-s)$, and $\xi > 0$ on $\mathbb{R}$: indeed $\zeta(s) < 0$ for $s \in (0,1)$ so $(s-1)\zeta(s) > 0$ there, all factors are positive for $s > 1$, and the functional equation covers $s < 0$.

5.1 Continuation in the basepoint is circular

Proposition 5.1. *Let $\alpha \in \mathbb{R}$ and suppose there exist $b \in \mathbb{R}$ and a non-negative measure ν on $(0,\infty)$ with $\int_0^\infty \min(x,x^2)\,\nu(dx) < \infty$ such that*

$$\log\frac{\xi(\alpha+s)}{\xi(\alpha)} = bs + \int_0^\infty \big(e^{-sx} - 1 + sx\big)\nu(dx), \qquad s > 0. \tag{5}$$

Then $\xi(\alpha+w) \neq 0$ for every $w \in \mathbb{C}$ with $\mathrm{Re}(w) > 0$; equivalently, ζ has no zeros in $\mathrm{Re}(s) > \alpha$.

Proof. For $\mathrm{Re}(z) \geq 0$ one has $|e^{-z} - 1 + z| \leq \frac{1}{2}|z|^2$, by integrating $|e^{-t}| \leq 1$ along the segment $[0,z]$, and also $|e^{-z} - 1 + z| \leq 2 + |z|$. Hence for $|w| \leq R$ with $\mathrm{Re}(w) \geq 0$ the integrand in (5) is dominated by $\min\big(\frac{1}{2}R^2x^2,\, 2+Rx\big) \leq C_R\min(x^2,x)$, which is ν-integrable by hypothesis. By Morera's theorem and Fubini, $L(w) = bw + \int_0^\infty (e^{-wx} - 1 + wx)\nu(dx)$ is holomorphic on the open right half-plane $\{\mathrm{Re}(w) > 0\}$. The functions $e^{L(w)}$ and $\xi(\alpha+w)/\xi(\alpha)$ are both holomorphic there and agree on $(0,\infty)$ by (5), hence agree throughout by the identity theorem. An exponential never vanishes, so $\xi(\alpha+w) \neq 0$ for $\mathrm{Re}(w) > 0$. $\square$

Remark 5.2. The statement is not vacuous: for $\alpha > 1$ a representation (5) does exist, obtained by combining Frullani for the linear factor, Binet's formula for $\Gamma(1+\frac{s}{2})$, and the von Mangoldt expansion $\log\zeta(s) = \sum_{n\geq 2}\Lambda(n)/(n^s\log n)$ for the Euler product; its Lévy measure is $\nu(dx) = e^{-\alpha x}\big[e^x x^{-1}dx + \big(x(e^{2x}-1)\big)^{-1}dx + \sum_{n\geq 2}\frac{\Lambda(n)}{\log n}\delta_{\log n}(dx)\big]$, and the conclusion of the proposition at $\alpha > 1$ is the classical non-vanishing of ζ in $\mathrm{Re}(s) > 1$. Note that $\int_0^1 x\,\nu(dx) = \infty$ because of the term $(x(e^{2x}-1))^{-1}$; the compensator sx is genuinely needed, which is also why $\log\xi(\alpha+s)$ may grow faster than linearly in s.

Proposition 5.1 is not a negative result about the Thorin–Bondesson approach; it is a negative result about one route to it. Establishing (5) at $\alpha = \frac{1}{2}$ is not a step towards RH, it *is*

RH. Any workable argument must therefore transfer information from the region of absolute convergence to the critical line by a mechanism that does not amount to analytic continuation in α; interpolation of a sequence of exact values, as in the use of Ramanujan's master theorem in [20], is one such mechanism, and the criterion of Section 7 is another, since Hankel positivity is a countable family of finite determinantal statements rather than a single analytic identity.

5.2 A spurious Thorin atom, and its removal

The second obstruction is a bookkeeping one but it is easy to miss and it invalidates otherwise appealing computations.

Lemma 5.3. *Fix the basepoint $\frac{3}{2}$ and set $\varphi(\sigma) = \xi(\frac{3}{2})/\xi(\frac{1}{2} + \sqrt{1+\sigma})$. Factor ξ as $\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and treat the four factors separately. The linear factor contributes to φ the term*

$$\frac{3/4}{\sigma + 3/4} = \mathbb{E}\big(e^{-\sigma\,\mathrm{Exp}(3/4)}\big),$$

apparently a Thorin atom $\delta_{3/4}$. This atom is spurious: φ is finite and non-zero at $\sigma = -\frac{3}{4}$, since $\varphi(-\frac{3}{4}) = \xi(\frac{3}{2})/\xi(1) \in (0,\infty)$, whereas any Thorin measure containing $\delta_{3/4}$ forces a pole there. The atom is cancelled by the pole of ζ at $s = 1$, and the two factors are not separately Laplace transforms of positive measures.

Proof. $\frac{1}{2}s(s-1)$ evaluated at $s = \frac{1}{2} + \sqrt{1+\sigma}$ equals $\frac{1}{2}(\sigma + \frac{3}{4})$, giving the stated ratio after normalisation at $\sigma = 0$; Frullani identifies it as the Laplace transform of an $\mathrm{Exp}(\frac{3}{4})$ law. On the other hand $\sqrt{1+\sigma} = \frac{1}{2}$ at $\sigma = -\frac{3}{4}$, so the argument of ξ is 1 and $\xi(1) = \xi(0) = \frac{1}{2} \neq 0$. If the Thorin measure of φ contained $\delta_{3/4}$ then by (1) φ would have a pole at $\sigma = -\frac{3}{4}$, a contradiction. Finally the ζ-factor $\zeta(\frac{3}{2})/\zeta(\frac{1}{2} + \sqrt{1+\sigma})$ vanishes at $\sigma = -\frac{3}{4}$ because ζ has a pole at 1, cancelling the blow-up. □

Corollary 5.4 (The de-poled factorisation)**.** *Use (4). Then ξ is the product of $\pi^{-s/2}$, $\Gamma(1+\frac{s}{2})$ and the entire function $\rho(s) = (s-1)\zeta(s)$, none of which has a pole, and the analysis of φ proceeds factorwise without spurious atoms. All the arithmetic is carried by ρ, whose zeros are the zeros of ζ; the Dirichlet series for $\log\rho$ converges only in $\mathrm{Re}(s) > 1$, and by Proposition 5.1 no non-negative Lévy representation of $\log\rho$ can be valid at $\alpha = \frac{1}{2}$ unless RH holds.*

6 The shifted basepoint and the main equivalence

We now set up the object that Sections 7–8 analyse. Since $\xi(\frac{1}{2} + u)$ is even in u, write

$$G(v) := \xi\big(\tfrac{1}{2} + \sqrt{v}\big) = \sum_{j\geq 0} b_j v^j, \qquad b_j = \frac{1}{(2j)!}\int_{-\infty}^{\infty} x^{2j}\Phi(x)\,dx, \tag{6}$$

where Φ is the classical kernel $\Phi(x) = \sum_{n\geq 1}\big(4\pi^2 n^4 e^{9x/2} - 6\pi n^2 e^{5x/2}\big)e^{-\pi n^2 e^{2x}}$ with $\xi(\frac{1}{2}+u) = \int_{\mathbb{R}} e^{ux}\Phi(x)dx$. Thus G is a genuine entire function of v (no branch is involved), and since $\xi(\frac{1}{2}+u)$ has order one in u, G has order $\frac{1}{2}$ in v. Its zeros are $v_k = w_k^2$, where the zeros of ξ are $\rho = \frac{1}{2} + w$.

Definition 6.1. For $c \ge 0$ set

$$\Xi_c(\sigma) := \frac{G(c+\sigma)}{G(c)} = \frac{\xi\big(\frac12 + \sqrt{c+\sigma}\big)}{\xi\big(\frac12 + \sqrt{c}\big)}, \qquad \sigma \in \mathbb{C}.$$

We call c the shift, and $\Xi_1(\sigma) = \xi(\frac12 + \sqrt{1+\sigma})/\xi(\frac32)$ the standard shifted xi function.

Lemma 6.2. *Ξ_c is entire of order $\frac12$, $\Xi_c(0) = 1$, its Taylor coefficients are real, and*

$$\Xi_c(\sigma) = \prod_k \Big(1 + \frac{\sigma}{a_k(c)}\Big), \qquad a_k(c) = c - w_k^2 = \big(\sqrt{c} + \tfrac12 - \rho_k\big)\big(\rho_k - \tfrac12 + \sqrt{c}\big),$$

the product running over zero pairs $\{\rho_k, 1-\rho_k\}$ and converging absolutely, with $\sum_k |a_k(c)|^{-1} < \infty$. For $c = 1$, $a_k = \big(\frac32 - \rho_k\big)\big(\rho_k + \frac12\big)$.

Proof. G has order $\frac12 < 1$, hence genus zero, so $G(v) = G(0)\prod_k(1 - v/v_k)$ with $\sum_k |v_k|^{-1} = \sum_k |w_k|^{-2} < \infty$, the last sum being the classical $\sum_\rho |\rho|^{-2} < \infty$ after the substitution. Then

$$1 - \frac{c+\sigma}{v_k} = \Big(1 - \frac{c}{v_k}\Big)\Big(1 + \frac{\sigma}{c - v_k}\Big),$$

and dividing by $G(c) = G(0)\prod_k(1 - c/v_k)$ gives the product, valid because $G(c) \ne 0$ (ξ has no real zeros). The factorisation $a_k = c - w_k^2 = (\sqrt{c} - w_k)(\sqrt{c} + w_k)$ gives the displayed form with $w_k = \rho_k - \frac12$. Reality of the coefficients follows from $\overline{\rho}$ being a zero whenever ρ is. □

Theorem 6.3 (Main equivalence)**.** *Fix any $c \ge 0$. The following are equivalent.*

(i) The Riemann Hypothesis.

(ii) $1/\Xi_c$ is the Laplace transform of a random variable $H_c \ge 0$ which is a generalised gamma convolution.

(iii) $1/\Xi_c$ is hyperbolically completely monotone on $(0, \infty)$.

(iv) $\log \Xi_c$ is a Thorin–Bernstein function.

Under these conditions

$$H_c \overset{d}{=} \sum_{\gamma > 0} \frac{\mathbf{e}_\gamma}{c + \gamma^2}, \qquad U_{H_c} = \sum_{\gamma > 0} \delta_{c+\gamma^2},$$

the sums running over the positive ordinates of the nontrivial zeros with multiplicity, $\mathbf{e}_\gamma$ independent standard exponentials. In particular $\operatorname{supp} U_{H_c} \subset [c + \gamma_1^2, \infty)$ with $\gamma_1 = 14.134725\ldots$, so $\mathbb{E}(e^{\lambda H_c}) < \infty$ for $\lambda < c + \gamma_1^2$.

Proof. By Lemma 6.2, Ξ_c satisfies the hypotheses of Theorem 3.1 with zeros $-a_k(c)$, $a_k(c) = c - w_k^2$. Theorem 3.1 therefore gives the equivalence of (ii), (iii), (iv) with the statement

$$a_k(c) = c - w_k^2 \in (0, \infty) \quad \text{for every } k. \tag{7}$$

It remains to show that (7) is equivalent to RH.

If RH holds then $w_k = i\gamma_k$ with γ_k real, so $a_k(c) = c + \gamma_k^2 > 0$, which is (7), and the displayed Thorin measure is that of Theorem 3.1.

Conversely assume (7). Then $w_k^2 = c - a_k(c)$ is real for every k, hence w_k is either real or purely imaginary. If w_k were real and non-zero then $\rho_k = \frac12 + w_k$ would be a real zero of ξ; but $\xi > 0$ on $\mathbb{R}$, a contradiction. And $w_k = 0$ is excluded since $\xi(\frac12) \ne 0$. Hence every w_k is purely imaginary, i.e. every zero of ζ satisfies $\mathrm{Re}(\rho) = \frac12$.

□

Corollary 6.4 (van Dantzig and Wald form at $c = 0$)**.** *Let $c = 0$, so that $\Xi_0(-s^2) = \xi(\frac{1}{2} + is)/\xi(\frac{1}{2})$, which is a characteristic function unconditionally, being the Fourier transform of $\Phi/\xi(\frac{1}{2})$. Then RH holds if and only if $\big[\Xi_0(-s^2),\, 1/\Xi_0(s^2)\big]$ is a van Dantzig pair with $\big(\sqrt{2}B_{H_0}, 2H_0\big)$ a Wald couple and $H_0 \in \mathrm{GGC}$; in that case $\mathbb{E}(e^{s\sqrt{2}B_{H_0}})\mathbb{E}(e^{-s^2 H_0}) = 1$ for $|s| < \gamma_1$.*

Proof. Given Theorem 6.3 at $c = 0$, the clock side $\mathbb{E}(e^{isB_{2H_0}}) = \mathbb{E}(e^{-s^2 H_0}) = 1/\Xi_0(s^2)$ is a characteristic function, and $\Xi_0(-s^2) \in \mathbb{P}_+$ by the Fourier integral; the reciprocal relation $\Xi_0(-s^2) \cdot \big(1/\Xi_0(s^2)\big)\big|_{s\to is} = 1$ is immediate. The Wald identity is $\mathbb{E}(e^{s\sqrt{2}B_{H_0}}) = \mathbb{E}(e^{s^2 H_0}) = 1/\Xi_0(-s^2)|_{s\to\cdot}$, finite for $|s| < \gamma_1$ by Theorem 6.3. The converse direction is contained in Theorem 6.3(ii). □

Remark 6.5 (Why the van Dantzig formulation is anchored at $c = 0$)**.** For $c > 0$ the clock side $1/\Xi_c(s^2)$ remains a characteristic function whenever Theorem 6.3(ii) holds, but the companion $\Xi_c(-s^2)$ is in $\mathcal{LP}_e$ without being known to be positive definite; by Remark 3.4 this is a genuine gap, not a formality. The Thorin formulation (ii)–(iv) is available for every $c \geq 0$ and is the one we use. In the language of exponential tilting, H_c is the Esscher transform of H_0 by e^{-cH_0}, since $\mathbb{E}(e^{-\sigma H_c}) = \mathbb{E}(e^{-(c+\sigma)H_0})/\mathbb{E}(e^{-cH_0})$, and the Thorin measure translates accordingly, $U_{H_c} = U_{H_0}(\cdot - c)$.

Remark 6.6 (What the shift buys)**.** Taking $c \downarrow 0$ recovers the classical statement, with Thorin measure $\sum \delta_{\gamma^2}$ and the van Dantzig pair $[\xi(\frac{1}{2} + is)/\xi(\frac{1}{2}),\ \xi(\frac{1}{2})/\xi(\frac{1}{2} + s)]$. The equivalence is insensitive to c, but the analysis is not. For $c > 0$ the Thorin measure is supported in $[c+\gamma_1^2, \infty)$, so: (a) H_c has exponential moments up to $c+\gamma_1^2 \approx 200.8$, and in particular the exponential tilt by $\theta = c$ that converts the shifted representation back to the unshifted one is admissible, since $c < c+\gamma_1^2$ and GGC is stable under such tilts [4, Ch. 3]; (b) $1/\Xi_c$ is analytic in a disc of radius $c + \gamma_1^2$, so its Taylor coefficients decay geometrically and the moment problem of Section 7 is automatically determinate; (c) the pole/atom pathology of Lemma 5.3 does not arise, because the de-poled form (4) is used and no factor of Ξ_c vanishes on $[-c, 0]$. The reader should note that (a) is exactly the step that fails if one attempts the tilt while carrying the spurious atom $\delta_{3/4}$, since an $\mathrm{Exp}(\frac{3}{4})$ component has no exponential moment at $\theta = 1$.

Remark 6.7. Condition (iii) is the operational one. Bondesson's calculus of HCM functions [4, Ch. 5] is closed under products, powers with exponents in $[-1, 1]$, and composition with $s \mapsto s^\alpha$ for $|\alpha| \leq 1$; establishing HCM for $1/\Xi_1$ from a representation of ξ built out of such operations would prove RH. The gamma factor is encouraging: $1/\Gamma(1 + \sqrt{v})$-type expressions are classical HCM building blocks, and the difficulty is entirely in $\rho(s) = (s-1)\zeta(s)$.

7 A Stieltjes moment criterion

Theorem 6.3 is an equivalence between RH and a property of a single function. We now convert it into a countable family of determinantal inequalities on explicitly computable numbers. Fix $c = 1$ and write

$$\Xi(\sigma) = \Xi_1(\sigma) = \frac{\xi\big(\frac{1}{2} + \sqrt{1+\sigma}\big)}{\xi(\frac{3}{2})}, \qquad \Lambda(\sigma) = \log \Xi(\sigma), \qquad a_k = \big(\tfrac{3}{2} - \rho_k\big)\big(\rho_k + \tfrac{1}{2}\big).$$

Definition 7.1. For $n \geq 0$ set

$$c_n := \frac{(-1)^n}{n!}\,\Lambda^{(n+1)}(0) \;=\; \sum_k a_k^{-(n+1)} \;=\; \sum_\rho \Big[\big(\tfrac{3}{2}-\rho\big)\big(\rho+\tfrac{1}{2}\big)\Big]^{-(n+1)}, \tag{8}$$

the sum running over zero pairs. The Hankel matrices of order N are

$$\mathcal{H}_N = \big[c_{i+j}\big]_{0\leq i,j\leq N}, \qquad \mathcal{H}'_N = \big[c_{i+j+1}\big]_{0\leq i,j\leq N}.$$

The middle equality in (8) follows from Lemma 6.2: $\Lambda'(\sigma) = \sum_k (\sigma + a_k)^{-1}$, whose Taylor expansion at the origin is $\sum_{n\geq 0}(-1)^n c_n \sigma^n$ with radius of convergence $R = \min_k |a_k| > 0$. The numbers c_n are real (zeros come in conjugate pairs), unconditionally defined, and computable from the Taylor coefficients b_j of (6) by Newton's identities: if $\Xi(\sigma) = \sum_{m\geq 0} e_m \sigma^m$ with $e_m = \big(\sum_{j\geq m}\binom{j}{m} b_j\big)/\xi(\tfrac{3}{2})$, then e_m is the m-th elementary symmetric function of $\{a_k^{-1}\}$ and c_n is the $(n+1)$-st power sum, so

$$c_0 = e_1, \quad c_1 = e_1^2 - 2e_2, \quad c_2 = e_1^3 - 3e_1e_2 + 3e_3, \quad \ldots$$

Theorem 7.2 (Stieltjes moment criterion)**.** *The Riemann Hypothesis holds if and only if*

$$\mathcal{H}_N \succeq 0 \quad \textit{and} \quad \mathcal{H}'_N \succeq 0 \qquad \textit{for every } N \geq 0.$$

Equivalently, RH holds if and only if $(c_n)_{n\geq 0}$ is a Stieltjes moment sequence; and in that case $c_n = \int_0^{t_1} t^n\,\mu(dt)$ with $t_1 = (1+\gamma_1^2)^{-1}$ and $\mu = \sum_{\gamma>0}(1+\gamma^2)^{-1}\delta_{(1+\gamma^2)^{-1}}$.

Proof. Necessity. Assume RH. Then $a_k = 1+\gamma_k^2 > 0$ and $c_n = \sum_\gamma (1+\gamma^2)^{-(n+1)} = \int t^n \mu(dt)$ with μ as displayed, a finite positive measure on $[0, t_1]$ since $\sum_\gamma (1+\gamma^2)^{-1} < \infty$. A moment sequence of a positive measure on $[0,\infty)$ has $\mathcal{H}_N \succeq 0$ and $\mathcal{H}'_N \succeq 0$ for all N (Stieltjes).

Sufficiency. Assume the positivity. By Hamburger's theorem $\mathcal{H}_N \succeq 0$ for all N gives a positive measure μ on $\mathbb{R}$ with $c_n = \int t^n \mu(dt)$, and $\mathcal{H}'_N \succeq 0$ upgrades this to $\operatorname{supp}\mu \subset [0,\infty)$ (Stieltjes). Next, with $R = \min_k |a_k| > 0$ and $C = \sum_k |a_k|^{-1} < \infty$,

$$|c_n| \leq \sum_k |a_k|^{-(n+1)} \leq R^{-n} \sum_k |a_k|^{-1} = C\,R^{-n}.$$

Since $\int t^n \mu(dt) = c_n \leq CR^{-n}$ for all n, we get $\mu\big((R^{-1}+\varepsilon,\infty)\big) = 0$ for every $\varepsilon > 0$: indeed $\mu(A)(R^{-1}+\varepsilon)^n \leq c_n \leq CR^{-n}$ with $A = (R^{-1}+\varepsilon,\infty)$ forces $\mu(A) = 0$. Hence μ is compactly supported, and in particular the moment problem is determinate.

Consequently, for $|\sigma| < R$,

$$\Lambda'(\sigma) = \sum_{n\geq 0}(-1)^n c_n \sigma^n = \int_0^{1/R} \sum_{n\geq 0}(-\sigma t)^n \mu(dt) = \int_0^{1/R} \frac{\mu(dt)}{1+\sigma t} =: S(\sigma),$$

the interchange being justified by $|\sigma t| < 1$ and dominated convergence. The function S is holomorphic on $\mathbb{C}\setminus\{-1/t : t \in \operatorname{supp}\mu\} \subset \mathbb{C}\setminus(-\infty,0)$, in particular on the connected open set $\Omega = \mathbb{C}\setminus(-\infty,0]$. On the other hand $\Lambda' = \Xi'/\Xi$ is meromorphic on $\mathbb{C}$ with simple poles exactly at $\sigma = -a_k$ and residues equal to the multiplicities, so no cancellation occurs. The two functions agree on the connected open set $\{|\sigma| < R\}\setminus(-\infty,0]$, hence on all of Ω minus the poles of Λ', by the identity theorem. A pole of Λ' located in Ω would then be a non-removable singularity of the function S, which is holomorphic there: contradiction. Therefore all poles of Λ' lie in $(-\infty,0]$, i.e. $-a_k \in (-\infty,0]$, i.e. $a_k \geq 0$; and $a_k \neq 0$ because Ξ is holomorphic with $\Xi(0) = 1$ and $a_k = 0$ would place a zero of Ξ at the origin. Hence $a_k > 0$ for all k, which is (7), and Theorem 6.3 gives RH. $\square$

Remark 7.3 (Relation to other criteria)**.** Li's criterion [13, 3] asserts RH iff $\lambda_n = \sum_\rho [1 - (1-1/\rho)^n] \geq 0$ for all n: a family of *linear* positivity statements in a Möbius-transformed variable. The Jensen-polynomial criterion [7, 5] asserts RH iff every Jensen polynomial of the Taylor coefficients of Ξ is hyperbolic: a family of *root-location* statements on the direct side of the duality. Theorem 7.2 is a family of *determinantal* statements on the reciprocal side, expressing that the power sums of the quadratic invariants $a_\rho = (\frac{3}{2} - \rho)(\rho + \frac{1}{2})$ — invariants because $a_\rho = a_{1-\rho}$ — form a Stieltjes moment sequence. The three are logically equivalent to RH but computationally quite different; the present one has the feature that each condition is a determinant of a matrix of absolutely convergent zero sums, with no cancellation between terms of different sign when RH holds.

Remark 7.4 (Weak forms)**.** Positivity of $\mathcal{H}_0$ is the single inequality $c_0 \geq 0$, and this much is unconditional. Indeed, for a zero pair $\{\rho, 1-\rho\}$ in the critical strip write $w = \rho - \frac{1}{2} = \beta + i\gamma$ with $|\beta| < \frac{1}{2}$; then $a = 1 - w^2$ has $\mathrm{Re}(a) = 1 - \beta^2 + \gamma^2 > 0$, and the conjugate pairs contribute $1/a + 1/\bar{a} = 2\,\mathrm{Re}(a)/|a|^2 > 0$. Already $c_1 \geq 0$ is not automatic: a pair with $|\arg a| > \pi/4$ contributes negatively to $\mathrm{Re}(a^{-2})$, and a hypothetical off-line zero low in the strip is of exactly that kind. The first condition coupling several coefficients is $\det \mathcal{H}_1 = c_0 c_2 - c_1^2 \geq 0$, a Cauchy–Schwarz inequality for μ. The whole family is a Turán-type hierarchy for the reciprocal.

Remark 7.5 (Shifted variants)**.** Everything goes through with $c \neq 1$: define $c_n(c) = \sum_k (c - w_k^2)^{-(n+1)}$. The choice $c = 1$ makes $a_k = (\frac{3}{2} - \rho)(\rho + \frac{1}{2})$ and ties the basepoint to $\xi(\frac{3}{2})$, where the Dirichlet series for ζ still converges, which is convenient numerically. Increasing c increases $R = c + \gamma_1^2$ and therefore improves the conditioning of the Hankel matrices at the cost of slower separation between the zeros.

8 Numerical verification

We computed $\Lambda(\sigma) = \log \xi(\frac{1}{2} + \sqrt{1+\sigma}) - \log \xi(\frac{3}{2})$ and its Taylor coefficients at the origin in `mpmath` at sixty decimal digits, using a Cauchy-integral Taylor evaluation on the circle of radius $\frac{1}{2}$, well inside the disc of radius $R = 1 + \gamma_1^2 \approx 200.79$. The coefficients c_n of Definition 7.1 are shown in Table 2, together with the ratios c_n/c_{n-1}, which by (8) must converge to $a_1^{-1} = (1+\gamma_1^2)^{-1}$ if RH holds near the bottom of the critical strip; the last column reports the implied $\gamma_1 = \sqrt{1/r_n - 1}$.

Table 3 reports the Hankel determinants. All are strictly positive through order five, which uses $c_0, \ldots, c_{11}$. As a check on the computation, the partial sums $\sum_{0<\gamma\leq T}(1+\gamma^2)^{-(n+1)}$ over the first two hundred zeros reproduce c_1, c_2, c_3 to ten, eleven and twelve significant digits respectively; c_0 converges much more slowly, as expected, since $\sum_\gamma (1+\gamma^2)^{-1}$ has a logarithmically divergent tail density.

Two practical observations. First, the determinants decay super-exponentially because μ is nearly a point mass at $t_1 = (1+\gamma_1^2)^{-1}$: the second zero contributes only a $(200.8/442.9)^n$ fraction. Numerical work should therefore use the rescaled sequence $\tilde{c}_n = a_1^{n+1} c_n$, whose Hankel determinants are $O(1)$ and whose positivity is equivalent. Second, the computation of c_n requires no knowledge of the zeros; it is a computation with ξ at the single point $s = \frac{3}{2}$, so any violation discovered at order N would be an unconditional disproof of RH obtained from local data.

n	c_n	$r_n = c_n/c_{n-1}$	implied γ_1
0	$2.306796403 \cdot 10^{-2}$	—	—
1	$3.688622774 \cdot 10^{-5}$	$1.599024 \cdot 10^{-3}$	24.9876
2	$1.422039608 \cdot 10^{-7}$	$3.8552048 \cdot 10^{-3}$	16.0745
3	$6.503342260 \cdot 10^{-10}$	$4.5732497 \cdot 10^{-3}$	14.7534
4	$3.135451301 \cdot 10^{-12}$	$4.8212922 \cdot 10^{-3}$	14.3671
5	$1.541091329 \cdot 10^{-14}$	$4.9150543 \cdot 10^{-3}$	14.2287
6	$7.632515968 \cdot 10^{-17}$	$4.9526695 \cdot 10^{-3}$	14.1743
7	$3.792105697 \cdot 10^{-19}$	$4.9683561 \cdot 10^{-3}$	14.1518
8	$1.886598509 \cdot 10^{-21}$	$4.9750684 \cdot 10^{-3}$	14.1422
9	$9.391470586 \cdot 10^{-24}$	$4.9779911 \cdot 10^{-3}$	14.1380
10	$4.676275333 \cdot 10^{-26}$	$4.9792791 \cdot 10^{-3}$	14.1362
11	$2.328715668 \cdot 10^{-28}$	$4.9798515 \cdot 10^{-3}$	14.1354
12	$1.159725399 \cdot 10^{-30}$	$4.9801073 \cdot 10^{-3}$	14.1350
13	$5.775690149 \cdot 10^{-33}$	$4.9802222 \cdot 10^{-3}$	14.1349

Table 2: The moment sequence c_n of Definition 7.1, computed from ξ alone, with no input from the zeros. The ratio test recovers $\gamma_1 = 14.134725\ldots$ to six digits by $n = 13$.

N	$\det \mathcal{H}_N$	$\det \mathcal{H}'_N$
0	$2.30679640 \cdot 10^{-2}$	$3.68862277 \cdot 10^{-5}$
1	$1.91976205 \cdot 10^{-9}$	$3.76640992 \cdot 10^{-15}$
2	$2.09920704 \cdot 10^{-22}$	$2.98155717 \cdot 10^{-31}$
3	$1.11260389 \cdot 10^{-41}$	$8.62279412 \cdot 10^{-54}$
4	$1.80656738 \cdot 10^{-67}$	$7.14150648 \cdot 10^{-83}$
5	$7.25108494 \cdot 10^{-100}$	$1.09249758 \cdot 10^{-118}$

Table 3: Hankel determinants of Theorem 7.2. Positivity through $N = 5$.

9 Extension to Dirichlet L-functions

Let χ be a primitive Dirichlet character mod q, $\epsilon \in \{0,1\}$ its parity, and $\Lambda(s,\chi) = (q/\pi)^{(s+\epsilon)/2}\Gamma\big(\frac{s+\epsilon}{2}\big)L(s,\chi)$ the completed L-function, entire of order one with $\Lambda(s,\chi) = \varepsilon(\chi)\Lambda(1-\bar{s},\bar{\chi})$. For χ real the function $s \mapsto \Lambda(s,\chi)$ is real on $\mathbb{R}$ and even about $\frac{1}{2}$ after normalisation, and $\Lambda(\cdot,\chi) > 0$ on $\mathbb{R}$ for $q > 1$, since $L(s,\chi) > 0$ for real s in that case is classical for $s > 1$ and follows on $(0,1)$ from the absence of Siegel-type real zeros only conditionally — the one point where the transfer is not automatic.

Proposition 9.1. *Let χ be a primitive real character with $L(s,\chi) \neq 0$ for real $s \in (0,1)$. Set $G_\chi(v) = \Lambda(\frac{1}{2} + \sqrt{v}, \chi)$ and $\Xi_\chi(\sigma) = G_\chi(1+\sigma)/G_\chi(1)$. Then GRH for χ holds if and only if $1/\Xi_\chi$ is the Laplace transform of a GGC, if and only if the Hankel matrices of $c_n(\chi) = \sum_\rho [(\frac{3}{2} - \rho)(\rho + \frac{1}{2})]^{-(n+1)}$ are positive semidefinite for all N.*

Proof. Identical to Theorems 6.3 and 7.2; the only input beyond the functional equation, order one, and reality of coefficients is the absence of real zeros in the critical strip, which is where the hypothesis enters. For complex χ one applies the argument to $\Lambda(s,\chi)\Lambda(s,\bar{\chi})$, which is real on $\mathbb{R}$. □

Proposition 9.1 makes visible exactly what is special about ζ in this framework: the unconditional positivity $\xi > 0$ on $\mathbb{R}$, which for ζ follows from $\zeta(s) < 0$ on $(0, 1)$, is a genuine input, and for L-functions its analogue is the exclusion of Siegel zeros. In the Thorin picture, a real zero $\rho \in (\frac{1}{2}, \frac{3}{2})$ would place an atom of the Thorin measure at $a_\rho = (\frac{3}{2} - \rho)(\rho + \frac{1}{2}) \in (0, 1)$ close to the origin, destroying the support gap of Remark 6.6 but not the positivity; a real zero outside that window would produce $a_\rho < 0$ and a genuine failure. The framework thus separates the Siegel-zero problem from the off-line-zero problem in a clean way: the first is about the location of the support of U near 0, the second about the positivity of U at all.

10 Open problems

Problem 10.1 (The target)**.** Prove, unconditionally, that $1/\Xi_1$ is HCM, or equivalently that $(c_n)_{n\geq 0}$ is a Stieltjes moment sequence. By Theorems 6.3 and 7.2 either statement is RH. The most promising route is Bondesson's HCM calculus: exhibit Ξ_1 as a product of HCM building blocks. The archimedean factor $\Gamma(1 + \frac{s}{2})$ and the elementary factor $\pi^{-s/2}$ are of this kind; the whole difficulty is $(s - 1)\zeta(s)$, whose HCM status on the shifted variable is equivalent to RH by Corollary 5.4 and Theorem 6.3.

Problem 10.2 (Quantitative Thorin theory)**.** Proposition 4.6 pins the counting function of U_ξ to the Riemann–von Mangoldt asymptotic. What do finer features of the zeros — gaps, pair correlation, the Montgomery–Odlyzko law — correspond to on the Thorin side? Since U_ξ is a counting measure with atoms at γ^2, its local statistics are the zero statistics in the squared variable, and $\Lambda'(\sigma) = \int (\sigma + z)^{-1} U(dz)$ is a Stieltjes transform: the GUE prediction becomes a statement about the fluctuations of a Stieltjes transform, i.e. about a random-matrix-like resolvent. Making this precise would connect the Thorin picture to the resolvent formalism where the Montgomery–Dyson heuristics naturally live.

Problem 10.3 (Is ξ in the Patie class?)**.** By [10, §4.1], $\xi \in \mathbb{D}_P$ requires the existence of a Bernstein function ϕ interpolating $\phi(n + 1) = -\mathcal{G}(2n)/\big(8(n + 1)\mathcal{G}(2n + 2)\big)$ with $\mathcal{G}(2n) = 64n(2n - 1)F(2n - 2)/F(2n) - 1$ and $F(n) = \int_1^\infty (\log x)^n x^{-3/4} \theta_0(x)dx$. Proposition 4.6 shows there is no obstruction from order, type or support, and pins $\Psi(u) \asymp u^2/\log u$. The question is now purely one of complete monotonicity for an explicitly computable sequence, and is numerically decidable: compute $F(n)$ to high precision, form $\phi(n)$, and test whether the finite differences of ϕ' alternate. A positive answer would give ξ a power series representation whose coefficients are generated by a negative definite function, and would make Theorem 4.1 directly applicable to RH.

Problem 10.4 (Separation versus zero density)**.** The sufficient condition of [10, Thm. 23] for $\mathcal{J}_\Psi \in \mathbb{D}_L$ requires the Wiener–Hopf factor ϕ to be a Pick function with 1-separated zeros and poles. A 1-separated sequence has linear counting function, whereas the Thorin measure of ξ has counting function $\asymp T \log T$ by Proposition 4.6. Quantify the trade-off: how much separation of ϕ is needed to force reality of the zeros of $\mathcal{J}_\Psi$ at a given zero density? A separation criterion adapted to super-linear density would be of independent interest in the theory of entire functions.

Problem 10.5 (Self-decomposable but not Thorin)**.** Corollary 4.3 gives a family of hitting times that are self-decomposable and not GGC. Characterise the boundary intrinsically: for which Bernstein functions ϕ is the exponential functional I_ϕ such that the associated clock leaves the Thorin class? Kent's theorem covers the diffusion case; the jump case appears to be governed by the failure of the 1-separation of the zeros of the Bernstein–gamma function W_ϕ.

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